PROPAGATION OF ERROR
Physics 230, Lab 1
Pre-Lab Assignment

Part I. One Variable

In most experiments, the quantity you are after is not the one you measure directly in the experiment. If you measure some quantity \( x \), you usually end up using \( x \) to compute a quantity \( q(x) \). When all the dust has cleared from your experiment and calculations, an important question usually arises: If the uncertainty (typically the standard deviation) of measurement \( x \) is \( s_x \), what is the uncertainty in the result \( q(x) \)? In this week’s laboratory we will develop a procedure by which we can answer this question. The fancy name for the process is propagation of error.

Let’s be very specific and consider an example. Suppose that I measure the length of one side of a cube and obtain the following data, all in cm:

\[
5.18, 5.31, 5.26, 5.16, 5.25, 5.14 \text{ cm.}
\]

The average of these measurements is 5.217 cm and the standard deviation is 0.027 cm. The final result would be

\[
5.22 \pm 0.03 \text{ cm.}
\]

For now, however, we will not round off to the proper number of significant figures so that we can see what’s going on.

Suppose that what I really want to know is the volume of the cube. I can handle that: \( V = (5.217)^3 = 142.0 \text{ cm}^3 \). But what is the uncertainty in my result? If the true lengths of each side of the cube were one standard deviation higher, the length would be 5.217 + 0.027 cm = 5.244 cm, and \( V \) would be \( (5.244 \text{ cm})^3 = 144.2 \text{ cm}^3 \). If the lengths were one standard deviation lower, the volume would be 139.8 cm \(^3\). We should then be comfortable saying that the volume is \( 142.0 \text{ cm}^3 \pm 2.2 \text{ cm}^3 \).

When we place a ± sign after a number, we are identifying the range within which the true answer probably lies. What we have done in the above example is to figure out how much variation in the volume is caused by a variation in the length. In formal terms, we are computing the variation in \( q(x) \) caused by a given variation in \( x \). We can always do this the long way as in the above example, but you probably have already figured out from the sound of the last sentence that a derivative might be in order. If the variations are not too large, it is much simpler to calculate the derivative of \( q(x) \) with respect to \( x \) in order to find the variation in \( q(x) \). The derivative is the change in \( q \) per change in \( x \). Thus, if \( s_x \) is the uncertainty or typically the standard deviation in the \( x \)-measurement and \( s_q \) is the uncertainty or standard deviation in \( q \)

\[
s_q = \frac{dq}{dx} s_x.
\]  

Let’s rework our example and employ eq. 1. We have \( q = x^3 \), so \( \frac{dq}{dx} = 3 x^2 \) and

\[
s_q = (3)(x^2)(s_x) = (3)(5.217 \text{ cm})^2(0.027 \text{ cm}) = 2.2 \text{ cm}^3
\]

This is the same result we obtained by the longer process. Let’s look at some special cases for the form of \( q(x) \):

\[
q = C x \Rightarrow s_q = C s_x
\]
\[ q = x^n \Rightarrow s_q = n x^{n-1} s_x \] (3)

\[ q = A \ln(bx) \Rightarrow s_q = \frac{A}{x} s_x \] (4)

\[ q = A e^{kx} \Rightarrow s_q = k A e^{kx} s_x \] (5)

**Example 1**

Suppose the measured value is \( x = 2.41 \pm 0.12 \) and we want to compute the uncertainties for:

a) \( u(x) = 1.5 \times \ln(4 \times x) = 1.5 \times \ln(4 \times 2.41) = 3.399 \)

b) \( v(x) = 5 e^{0.3x} = 5 e^{0.3(2.41)} = 10.30 \)

**Solution:**

a) From eq. 4 we have

\[ s_u = \left( \frac{1.5}{2.41} \right) \times 0.12 = 0.075 \]

Hence, \( u = 3.40 \pm 0.08 \)

b) From eq. 5 we have

\[ s_v = (0.3) \times (5) \times e^{(0.3)(2.41)} \times 0.12 = 0.37 \]

Hence, \( v = 10.3 \pm 0.4 \)

When you do this using real data, don’t forget the units!

You can see on your own how the above rules can be combined and extended to deal with more complicated situations

**Part II. Multiple variables**

In most real applications of the propagation of error, the quantity \( q \) to be calculated depends upon several variables \( x, y, z \ldots \) The formula for the uncertainty in \( q \) from the uncertainties in \( x, y, z, \ldots \) looks formidable, but it is not hard to see where it comes from. First of all, we recall that eq. 1 describes the error in \( q \) that is produced by one variable (say, \( x \)). If we call this error \( s_{q,x} \), then

\[ s_{q,x} = \frac{\partial q}{\partial x} s_x \]

Recall that \( \frac{\partial q}{\partial x} \) is the partial derivative of \( q \) with respect to \( x \). This means that we take the derivative with respect to \( x \) while holding all the other variables constant.

The uncertainty in \( q \) is just some combination of all the uncertainties \( s_{q,x}, s_{q,y}, \ldots \) due to all the different variables that determine \( q \). Often, one’s first impulse is to simply add the uncertainties:
\[ s_q = s_{q,x} + s_{q,y} + \ldots \quad \text{(THIS IS WRONG!!).} \]

(Yes, I know that's what you've been taught to do in other courses. It's still wrong. Don't do it any more. Under any circumstances.) The trouble here is that \( x, y, \ldots \) are independent and simply adding the uncertainties does not take into account that random error contributions from each variable are just as likely to subtract as they are to add. The idea of independence is important. The idea is that errors in one variable do not influence errors in the other variables. This is typically the case. If you consider a determination of density \( \rho \) where \( \rho = \frac{m}{V} \), we would not expect errors in \( m \), usually determined by weighing, to affect errors in \( V \), usually determined dimensional measurement. If the errors affect each other, it’s time to enlist the services of a statistician. The correct expression for the combination of independent errors turns out to be the square root of the sum of the squares:

\[
 s_q = \sqrt{\left(\frac{\partial q}{\partial x} s_x\right)^2 + \left(\frac{\partial q}{\partial y} s_y\right)^2 + \left(\frac{\partial q}{\partial z} s_z\right)^2 + \ldots} \quad (6)
\]

**Example 2**

Suppose we have measured the distance \( t \) for an object to fall a distance \( y \) and wish to compute the acceleration of gravity from

\[ g = \frac{2 y}{t^2} \]

Moreover, suppose that your results are \( y = 24.8 \pm 0.4 \text{ cm} \) and \( t = 0.224 \pm 0.004 \text{ s} \). Then

\[
 g = \frac{(2)(24.8 \text{ cm})}{(0.224 \text{ s})^2} = 988.5 \frac{\text{cm}}{\text{s}^2}.
\]

Now

\[
 \frac{\partial g}{\partial y} = \frac{2}{t^2} \frac{2}{(0.224 \text{ s})^2} = 39.9 \text{ s}^{-2},
\]

\[
 \frac{\partial g}{\partial t} = -4 \frac{y}{t^3} = -\frac{(4)(24.8 \text{ cm})}{(0.224 \text{ s})^3} = -8826 \frac{\text{cm}}{\text{s}^3}.
\]

We are able to ignore the signs because we will be squaring the negative terms.

\[
 s_g = \sqrt{\left(39.9 \text{ s}^{-2}\right)(0.4 \text{ cm})^2 + \left(8826 \frac{\text{cm}}{\text{s}^3}\right)(0.004 \text{ s})^2} = 39 \frac{\text{cm}}{\text{s}^2}.
\]

After rounding, our final result would read \( g = 990 \pm 40 \frac{\text{cm}}{\text{s}^2} \).

**IMPORTANT NOTE:** It is usually a good idea to round off only after all the smoke of the calculations has cleared. For the uncertainty, it is often best to round to one significant figure. Also, never round the uncertainty limits downward.

**Special cases**

There are several special cases worth working through. They all follow from Eq. 6.

A. Sum and difference:

For \( q = x + y \) or \( q = x - y \)

\[
 s_q = \sqrt{s_x^2 + s_y^2}.
\]

(7)

For a linear sum \( q = a_1 x_1 + a_2 x_2 + \ldots \)

\[
 s_q = \sqrt{a_1^2 s_1^2 + a_2^2 s_2^2 + \ldots}.
\]

(8)
Example 3
In an experiment you have measured \( x = 12.57 \pm 0.14 \text{ cm} \) and \( y = 5.98 \pm 0.09 \text{ cm} \) and you want \( q = x - 2y = 0.61 \text{ cm} \). With \( a_1 = 1 \) and \( a_2 = -2 \), eq. 8 gives
\[
s_q = \sqrt{(0.14 \text{ cm})^2 + (-2)(0.09 \text{ cm})^2} = 0.23 \text{ cm}.
\]
and the answer would be \( q = 0.6 \pm 0.2 \text{ cm} \). Notice the relatively large uncertainty that results from taking the (small) difference between two comparable numbers.

B. Product and quotient

For \( q = xy \) or \( q = \frac{x}{y} \), it is not hard to show that
\[
s_q = q \sqrt{s_x^2 + s_y^2}
\]
and
\[
s_q = q \sqrt{n_1^2 \left(\frac{s_x}{x_1}\right)^2 + n_2^2 \left(\frac{s_x}{x_2}\right)^2 + \ldots}
\]

Example 4
Let’s rework our calculation of \( g \) from above. It has the form \( g = 2y t^{-2} \), so \( y \) and \( t \) take the place of \( x \) and \( y \) in eq. 10. The measured values are
\( y = 24.8 \pm 0.4 \text{ cm} \) and \( t = 0.224 \pm 0.004 \text{ s} \), which gave \( g = 988.5 \text{ cm/s}^2 \). In our present example \( m = 1 \) and \( n = -2 \) so
\[
s_q = \left(\frac{998.5 \text{ cm}}{\text{s}^2}\right) \sqrt{\left(1 \right)^2 \left(\frac{0.4 \text{ cm}}{24.8 \text{ cm}}\right)^2 + (-2)^2 \left(\frac{0.004 \text{ s}}{0.224 \text{ s}}\right)^2} = 39 \text{ cm/s}^2.
\]

C. Rectangular to Polar Conversion (also known as the radar problem)
Suppose that you have measured the components of a vector (see the drawing below) and you want its magnitude and direction. The familiar formulas are:
\[
r = \sqrt{x^2 + y^2}, \quad \tan(\theta) = \frac{y}{x}.
\]
(Note that \( \theta \) is in RADIANS!)

You should be able to show that
\[
s_r = \left(\frac{1}{r}\right) \sqrt{(x s_x)^2 + (y s_y)^2},
\]
\[
s_\theta = \left(\frac{1}{r^2}\right) \sqrt{(y s_x)^2 + (x s_y)^2}.
\]
Example 5
You know that the location of a certain town is 40 miles north and 26 miles east of your starting point, each with an uncertainty of 0.5 miles. You want to compute its distance and bearing.

Now, \[ r = \sqrt{(26 \text{ mi})^2 + (40 \text{ mi})^2} = 47.7 \text{ mi}, \]
\[ \theta = \tan^{-1}\left(\frac{40}{26}\right) = 0.994 \text{ rad}. \]

Hence \[ s_r = \left(\frac{1}{47.7 \text{ mi}}\right) \sqrt{[(26 \text{ mi})(0.5 \text{ mi})]^2 + [(40 \text{ mi})(0.5 \text{ mi})]^2} = 0.5 \text{ mi}, \]
\[ s_\theta = \left(\frac{1}{47.7 \text{ mi}}\right)^2 \sqrt{[(40 \text{ mi})(0.5 \text{ mi})]^2 + [(26 \text{ mi})(0.5 \text{ mi})]^2} = 0.0105 \text{ rad}. \]

And we have
\[ r = 47.7 \pm 0.5 \text{ mi}, \]
\[ \theta = 0.994 \pm 0.011 \text{ rad} = 57.0^\circ \pm 0.6^\circ. \]

This ends the special cases that you are most likely to run into. If you do have to deal with a more complicated situation, just remember that you can apply eq. 6 in stages: combine two of your variables into a third, compute its uncertainty, combine it with one or more other variables, compute its uncertainty, and so on. Often, you will be able to use the special case rules along the way.

Also, notice something else that you can learn from the structure of eq 6. Each of the individual terms that are being combined is the contribution of one of the variables to the standard error of the result. In practice, it often happens that one of these terms is much bigger than the others. If this is the case, it tells you two things: first, that just figuring that one term - using eq. 1 instead of the messier eq. 6 - will do, at least as a quick approximation; and second, that if you want to improve the experiment, which of your measurements you need to work on.